

# A new multi-component CKP hierarchy <sup>\*†</sup>

Hongxia Wu

1. *Department of Mathematics, Jimei University, Xiamen, 361021, China*

2. *Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China*

Xiaojun Liu

*Department of Mathematics, Chinese Agriculture University, Beijing, PR China*

Yunbo Zeng

*Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China*

## Abstract

We construct a new multi-component CKP hierarchy based on the eigenfunction symmetry reduction. It contains two types of CKP equation with self-consistent sources which Lax representations are presented. Also it admits reductions to  $k$ -constrained CKP hierarchy and to a  $(1+1)$ -dimensional soliton hierarchy with self-consistent source, which include two types of Kaup-Kuperschmidt equation with self-consistent sources and of bi-directional Kaup-Kuperschmidt equation with self-consistent sources.

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## 1. Introduction

Multi-component KP hierarchy attract a lot of interests from both physical and mathematical points of view [1-8]. The multi-component KP hierarchy given in [1] contains many physically relevant nonlinear integrable systems such as Davey-Stewartson equation, two-dimensional Toda lattice and three-wave resonant interaction ones. Another kind of multi-component KP equation is the so-called KP equation with self-consistent sources, which was initiated by V.K. Mel'nikov [9-11]. The first type of KP equation with self-consistent sources (KPSCS) arises in some physical modes describing the interaction of long and short wave [8-10,12], and the second type of KPSCS is presented in [8,11,13]. Recently a method was proposed in [8] to construct a new multi-component KP hierarchy which includes first and second type of KPSCS. However, little attention has been paid to the multi-component CKP hierarchy. In addition, the CKP equation with self-consistent sources has not been found out yet.

It is known that the Lax equation of KP hierarchy is given by [14]

$$L_{t_n} = [B_n, L] \quad (1.1)$$

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\* Corresponding author: +8610 68916895

† E-mail address: wuhongxia@bit.edu.cn

where

$$L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \dots \quad (1.2)$$

is pseudo-differential operator,  $\partial$  denotes  $\partial/\partial_x$ ,  $u_i$ ,  $i = 1, 2, \dots$ , are functions in infinitely many variables  $t = (t_1, t_2, t_3, \dots)$  with  $t_1 = x$ , and  $B_n = L_+^n$  stands for the differential part of  $L^n$ .

Owing to the commutativity of  $\partial_{t_n}$  flows, we obtain zero-curvature equations of KP hierarchy

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0 \quad (1.3)$$

Eigenfunction  $\Phi$  (adjoint eigenfunction  $\Phi^*$ ) satisfy the linear evolution equations

$$\Phi_{t_n} = B_n(\Phi) \quad (\Phi_{t_n}^* = -B_n^*(\Phi^*)) \quad (1.4)$$

The compatibility condition of (1.4) is exactly (1.3).

The CKP hierarchy [15] is obtained from the KP hierarchy by ignoring the time variables  $t_2, t_4, t_6, \dots$  (i.e. including only the odd time variables  $t_3, t_5, t_7, \dots$ ) and by imposing at the same time the following antisymmetry condition on the KP Lax operator

$$L + L^* = 0 \quad (1.5)$$

It follows immediately from (1.5) that

$$u_2 = -\frac{1}{2}u_1', u_4 = -\frac{3}{2}u_3' + \frac{1}{4}u_1^{(3)}, \dots$$

and  $\Phi = \Phi^*$ ,  $B_n = -B_n^*$  for  $n$  odd. Taking  $n = 3, k = 5$ , (1.3) and (1.5) lead to the CKP equation

$$u_{t_5} - \frac{5}{9}u_{t_3}^{(2)} - \frac{5}{3}uu_{t_3} - \frac{5}{9}\partial_x^{-1}u_{t_3 t_3} + \frac{1}{9}u^{(5)} + \frac{25}{6}u'u^{(2)} + \frac{5}{3}uu^{(3)} - \frac{5}{3}u'\partial_x^{-1}u_{t_3} + 5u^2u' = 0 \quad (1.6)$$

where we use the notation  $u^{(i)} = \frac{\partial^i}{\partial x^i}$  in this paper.

In this paper, following the idea in [8] and using the eigenfunction symmetry constraint, we firstly introduce a new type of Lax equations which consist of the new time  $\tau_k$ -flow and the evolutions of wave functions. Under the evolutions of wave functions, the commutativity of the evolutions of  $\tau_k$ -flow and  $t_n$ -flow gives rise to a new multi-component CKP (mcCKP) hierarchy. This hierarchy enables us to obtain the first and the second types of CKP equation with self-consistent sources (CKPSCS) and their Lax representations directly. This implies that the new mcCKP hierarchy can be regarded as CKP hierarchy with self-consistent sources (CKPHSCS). Moreover, this new mcCKP hierarchy can be reduced to two integrable hierarchies: a (1+1)-dimensional soliton hierarchy with self-consistent source and the  $k$ -constrained CKP hierarchy ( $k$ -CKPH), which contain the first type and the second type of Kaup-Kuperschmidt equation with self-consistent sources and of bi-directional Kaup-Kuperschmidt equation with self-consistent sources, respectively. Thus, the new mcCKP hierarchy provides an effective way to find (1+1)-dimensional and (2+1)-dimensional soliton equations with self-consistent sources as well as their Lax representations. Our paper is organized as follows. In section 2, we construct the new mcCKP hierarchy and show that it contains the first and the second types of CKPSCS. In section 3, the mcCKP hierarchy is reduced to a (1+1)-dimensional soliton hierarchy with self-consistent source and the  $k$ -constrained CKP hierarchy, respectively. In section 4, some conclusions are given.

## 2. New multi-component CKP hierarchy

Following the idea in [8] and using the eigenfunction symmetry constraint for CKP hierarchy [16], we define  $\tilde{B}_k$  by

$$\tilde{B}_k = B_k + \sum_{i=1}^N (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i) \quad (2.1)$$

where  $q_i, r_i$  satisfy (1.4). Then we may introduce a new Lax equation given by

$$L_{\tau_k} = [B_k + \sum_{i=1}^N (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i), L] \quad (2.2a)$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = B_n(r_i), \quad i = 1, \dots, N \quad (2.2b)$$

where  $n, k$  are odd.

**Lemma 1**  $[B_n, r \partial^{-1} q + q \partial^{-1} r]_- = (r \partial^{-1} q + q \partial^{-1} r)_{t_n}$

**Proof:** Set  $B_n = \sum_{i=1}^n a_i \partial^i$ . Then we have

$$\begin{aligned} [B_n, r \partial^{-1} q + q \partial^{-1} r]_- &= \sum_{i=1}^n (a_i r^{(i)} \partial^{-1} q + a_i q^{(i)} \partial^{-1} r) - \sum_{i=1}^n (r \partial^{-1} q a_i \partial^i + q \partial^{-1} r a_i \partial^i)_- \\ &= B_n(r) \partial^{-1} q + B_n(q) \partial^{-1} r - \sum_{i=1}^n (r \partial^{-1} q a_i \partial^i + q \partial^{-1} r a_i \partial^i)_- \end{aligned}$$

Applying integration by parts to the second term

$$\sum_{i=1}^n (r \partial^{-1} q a_i \partial^i + q \partial^{-1} r a_i \partial^i)_- = \dots = \sum_{i=1}^n (-1)^i [r \partial^{-1} (a_i q)^{(i)} + q \partial^{-1} (a_i r)^{(i)}] = r \partial^{-1} B_n^*(q) + q \partial^{-1} B_n^*(r)$$

Noticing the facts that  $q^* = q, r^* = r, q_{t_n}^* = -B_n^*(q^*)$  and  $r_{t_n}^* = -B_n^*(r^*)$ , we can complete the proof immediately.

**Theorem 1.** The commutativity of (1.1) and (2.2a) under (2.2b) leads to the following new integrable multi-component CKP (mcCKP) hierarchy

$$B_{n,\tau_k} - (B_k + \sum_{i=1}^N (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i))_{t_n} + [B_n, B_k + \sum_{i=1}^N (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i)] = 0 \quad (2.3a)$$

or equivalently

$$B_{n,\tau_k} - B_{k,t_n} + [B_n, B_k] + \sum_{i=1}^N \{ [B_n, r_i \partial^{-1} q_i + q_i \partial^{-1} r_i] - B_n(r_i) \partial^{-1} q_i \quad (2.3a')$$

$$- r_i \partial^{-1} B_n(q_i) - B_n(q_i) \partial^{-1} r_i - q_i \partial^{-1} B_n(r_i) \} = 0$$

$$q_{i,t_n} = B_n(q_i), \quad r_{i,t_n} = B_n(r_i), \quad i = 1, \dots, N \quad (2.3b)$$

where  $n$  and  $k$  are odd. Under (2.3b), the Lax pair for (2.3a) is given by

$$\psi_{t_n} = B_n(\psi), \quad \psi_{\tau_k} = [B_k + \sum_{i=1}^N (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i)](\psi) \quad (2.4)$$

**Proof:** We will show that under (2.3b), (1.1) and (2.2a) lead to (2.3a). For convenience, we assume  $N = 1$  and denote  $q_1, r_1$  by  $q, r$ . By (1.1), (2.2) and lemma 1, we have

$$\begin{aligned}
B_{n,\tau_k} &= (L_{\tau_k}^n)_+ = [B_k + r\partial^{-1}q + q\partial^{-1}r, L^n]_+ = [B_k + r\partial^{-1}q + q\partial^{-1}r, L_+^n]_+ + [B_k + r\partial^{-1}q + q\partial^{-1}r, L_-^n]_+ \\
&= [B_k + r\partial^{-1}q + q\partial^{-1}r, L_+^n] - [B_k + r\partial^{-1}q + q\partial^{-1}r, L_+^n]_- + [B_k, L_-^n]_+ \\
&= [B_k + r\partial^{-1}q + q\partial^{-1}r, B_n] - [r\partial^{-1}q + q\partial^{-1}r, B_n]_- + [B_n, L^k]_+ \\
&= [B_k + r\partial^{-1}q + q\partial^{-1}r, B_n] + (r\partial^{-1}q + q\partial^{-1}r)_{t_n} + (B_k)_{t_n} \\
&= [B_k + r\partial^{-1}q + q\partial^{-1}r, B_n] + (B_k + r\partial^{-1}q + q\partial^{-1}r)_{t_n}
\end{aligned}$$

**Remark 1.** (2.3a') and (2.4) indicate that the mcCKP hierarchy can be regarded as the CKP hierarchy with self-consistent sources and is Lax integrable.

We now list some equations in this new mcCKP hierarchy.

**Example 1** (The first type of CKPSCS) For  $n = 3, k = 5$ , (2.3) with  $u = u_1$  leads to the first type of the CKP equation with self-consistent sources

$$\begin{aligned}
u_{\tau_5} - \frac{5}{9}u_{t_3}^{(2)} - \frac{5}{3}uu_{t_3} - \frac{5}{9}\partial_x^{-1}u_{t_3t_3} + \frac{1}{9}u^{(5)} + \frac{25}{6}u'u^{(2)} + \frac{5}{3}uu^{(3)} - \frac{5}{3}u'\partial_x^{-1}u_{t_3} + 5u^2u' + 2\sum_{i=1}^N(q_i'r_i + q_i r_i') &= 0, \\
q_{i,t_3} = q_i^{(3)} + 3uq_i' + \frac{3}{2}u'q_i, \quad r_{i,t_3} = r_i^{(3)} + 3ur_i' + \frac{3}{2}u'r_i, \quad i = 1, \dots, N
\end{aligned} \tag{2.5}$$

The Lax pair of (2.5) is given by

$$\begin{aligned}
\psi_{t_3} &= (\partial^3 + 3u\partial + \frac{3}{2}u')(\psi), \\
\psi_{\tau_5} &= (\partial^5 + 5u\partial^3 + \frac{15}{2}u'\partial^2 + (\frac{5}{3}\partial_x^{-1}u_{t_3} + \frac{35}{6}u^{(2)} + 5u^2)\partial + [\frac{5}{6}u_{t_3} + \frac{5}{3}u^{(3)} + 5uu' + \sum_{i=1}^N(q_i\partial^{-1}r_i + r_i\partial^{-1}q_i)])(\psi)
\end{aligned} \tag{2.6}$$

**Example 2 (The second type of CKPSCS)** For  $n = 5, k = 3$ , (2.3) with  $u_1 = u$  yields the second type of CKP equation with self-consistent sources

$$\begin{aligned}
u_{t_5} - \frac{5}{9}u_{\tau_3}^{(2)} - \frac{5}{3}uu_{\tau_3} - \frac{5}{9}\partial_x^{-1}u_{\tau_3\tau_3} + \frac{1}{9}u^{(5)} + \frac{25}{6}u'u^{(2)} + \frac{5}{3}uu^{(3)} - \frac{5}{3}u'\partial_x^{-1}u_{\tau_3} + 5u^2u' &= \\
\frac{1}{3}\sum_{i=1}^N[\frac{10}{3}(q_i r_i)_{\tau_3} + \frac{20}{3}q_i^{(3)}r_i + \frac{20}{3}r_i^{(3)}q_i + 10q_i^{(2)}r_i' + 10r_i^{(2)}q_i' + 20uq_i'r_i + 20uq_i r_i' + 20u'q_i r_i], \\
q_{i,t_5} = q_i^{(5)} + 5uq_i^{(3)} + \frac{15}{2}u'q_i^{(2)} + (\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{35}{6}u^{(2)} + 5u^2 + \frac{10}{3}\sum_{i=1}^N q_i r_i)q_i' + [\frac{5}{6}u_{\tau_3} + \frac{5}{3}u^{(3)} + 5uu' + \frac{5}{3}\sum_{i=1}^N (q_i r_i)']q_i, \\
r_{i,t_5} = r_i^{(5)} + 5ur_i^{(3)} + \frac{15}{2}u'r_i^{(2)} + (\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{35}{6}u^{(2)} + 5u^2 + \frac{10}{3}\sum_{i=1}^N q_i r_i)r_i' + [\frac{5}{6}u_{\tau_3} + \frac{5}{3}u^{(3)} + 5uu' + \frac{5}{3}\sum_{i=1}^N (q_i r_i)']r_i, \\
i = 1, \dots, N
\end{aligned} \tag{2.7}$$

The Lax pair of (2.7) is given by

$$\begin{aligned}\psi_{\tau_3} &= [\partial^3 + 3u\partial + \frac{3}{2}u' + \sum_{i=1}^N (q_i\partial^{-1}r_i + r_i\partial^{-1}q_i)](\psi), \\ \psi_{t_5} &= (\partial^5 + 5u\partial^3 + \frac{15}{2}u'\partial^2 + (\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{35}{6}u^{(2)} + 5u^2 + \frac{10}{3}\sum_{i=1}^N q_i r_i)\partial + [\frac{5}{6}u_{\tau_3} + \frac{5}{3}u^{(3)} + 5uu' + \frac{5}{3}\sum_{i=1}^N (q_i r_i)']) (\psi)\end{aligned}\quad (2.8)$$

### 3. The $n$ - reduction and $k$ - constraint of (2.3)

#### 3.1 The $n$ - reduction of (2.3)

The  $n$ - reduction of (2.3) is given by [14]

$$L^n = B_n, \quad \text{or} \quad L_-^n = 0 \quad (2.9)$$

which implies that

$$L_{t_n} = [B_n, L] = [L^n, L] = 0, \quad B_{k,t_n} = (L_+^k)_{t_n} = 0, \quad \text{and} \quad q_{i,t_n} = r_{i,t_n} = 0 \quad (2.10)$$

If  $q_i$  and  $r_i$  are wave function, they have to satisfy [14]

$$B_n(q_i) = L^n(q_i) = \lambda_i^n q_i, \quad B_n(r_i) = L^n(r_i) = \lambda_i^n r_i \quad (2.11)$$

So it is reasonable to impose the relation (2.11) in the  $n$ - reduction case. By using the Lemma 1 and (2.10), we can conclude that the constraint (2.9) is invariant under the  $\tau_k$ - flow. Due to (2.10) and (2.11), one can drop  $t_n$ - dependency from (2.3) and get the following (1+1)-dimensional integrable hierarchy with self-consistent sources

$$\begin{aligned}B_{n,\tau_k} + [B_n, B_k + \sum_{i=1}^N (q_i\partial^{-1}r_i + r_i\partial^{-1}q_i)] &= 0, \\ B_n(q_i) = \lambda_i^n q_i, \quad B_n(r_i) = \lambda_i^n r_i, \quad i &= 1, \dots, N\end{aligned}\quad (2.12)$$

with the Lax pair given by

$$\begin{aligned}B_n(\psi) &= \lambda^n \psi, \\ \psi_{\tau_k} &= [B_k + \sum_{i=1}^N (q_i\partial^{-1}r_i + r_i\partial^{-1}q_i)](\psi)\end{aligned}\quad (2.13)$$

**Example 3 (The first type of KKESCS)** For  $n = 3, k = 5$ , (2.12) presents the first type of Kaup-Kuperschmidt equation with self-consistent sources

$$\begin{aligned}u_{\tau_5} + \frac{1}{9}u^{(5)} + \frac{25}{6}u'u^{(2)} + \frac{5}{3}uu^{(3)} + 5u^2u' + 2\sum_{i=1}^N (q_i'r_i + q_i r_i') &= 0, \\ q_i^{(3)} + 3uq_i' + \frac{3}{2}u'q_i &= \lambda_i^3 q_i \\ r_i^{(3)} + 3ur_i' + \frac{3}{2}u'r_i &= \lambda_i^3 r_i, \quad i = 1, \dots, N\end{aligned}\quad (2.14)$$

(2.13) with  $n = 3, k = 5$  leads to the Lax pair of (2.14)

$$\begin{aligned} (\partial^3 + 3u\partial + \frac{3}{2}u')(\psi) &= \lambda\psi, \\ \psi_{\tau_5} &= [\partial^5 + 5u\partial^3 + \frac{15}{2}u'\partial^2 + (\frac{35}{6}u^{(2)} + 5u^2)\partial + (\frac{5}{3}u^{(3)} + 5uu') + \sum_{i=1}^N (q_i\partial^{-1}r_i + r_i\partial^{-1}q_i)](\psi) \end{aligned} \quad (2.15)$$

If we take  $q_i = r_i = 0$ , then (2.14) reduces to the Kaup-Kuperschmidt equation [17].

**Example 4 (The first type of BDKKESCS)** For  $n = 5, k = 3$ , (2.12) presents the first type of bi-directional Kaup-Kuperschmidt equation with self-consistent sources

$$\begin{aligned} & -\frac{5}{9}u_{\tau_3}^{(2)} - \frac{5}{3}uu_{\tau_3} - \frac{5}{9}\partial_x^{-1}u_{\tau_3\tau_3} + \frac{1}{9}u^{(5)} + \frac{25}{6}u'u^{(2)} + \frac{5}{3}uu^{(3)} - \frac{5}{3}u'\partial_x^{-1}u_{\tau_3} \\ & + 5u^2u' = \frac{1}{3}\sum_{i=1}^N [\frac{10}{3}(q_i r_i)_{\tau_3} + \frac{20}{3}q_i^{(3)}r_i + \frac{20}{3}r_i^{(3)}q_i + 10q_i^{(2)}r'_i + 10r_i^{(2)}q'_i + 20uq'_i r_i + 20uq_i r'_i + 20u'q_i r_i], \\ & q_i^{(5)} + 5uq_i^{(3)} + \frac{15}{2}u'q_i^{(2)} + (\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{35}{6}u^{(2)} + 5u^2 + \frac{10}{3}\sum_{i=1}^N q_i r_i)q'_i + [\frac{5}{6}u_{\tau_3} + \frac{5}{3}u^{(3)} + 5uu' + \frac{5}{3}\sum_{i=1}^N (q_i r_i)']q_i = \lambda_i^5 q_i, \\ & r_i^{(5)} + 5ur_i^{(3)} + \frac{15}{2}u'r_i^{(2)} + (\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{35}{6}u^{(2)} + 5u^2 + \frac{10}{3}\sum_{i=1}^N q_i r_i)r'_i + \\ & [\frac{5}{6}u_{\tau_3} + \frac{5}{3}u^{(3)} + 5uu' + \frac{5}{3}\sum_{i=1}^N (q_i r_i)']r_i = \lambda_i^5 r_i, i = 1, \dots, N \end{aligned} \quad (2.16)$$

with the Lax pair given by

$$\begin{aligned} \psi_{\tau_3} &= [\partial^3 + 3u\partial + \frac{3}{2}u' + \sum_{i=1}^N (q_i\partial^{-1}r_i + r_i\partial^{-1}q_i)](\psi), \\ \{\partial^5 + 5u\partial^3 + \frac{15}{2}u'\partial^2 + (\frac{5}{3}\partial_x^{-1}u_{\tau_3} + \frac{35}{6}u^{(2)} + 5u^2 + \frac{10}{3}\sum_{i=1}^N q_i r_i)\partial + \\ & [\frac{5}{6}u_{\tau_3} + \frac{5}{3}u^{(3)} + 5uu' + \frac{5}{3}\sum_{i=1}^N (q_i r_i)']\}(\psi) = \lambda^5 \psi \end{aligned} \quad (2.17)$$

If we take  $q_i = r_i = 0$ , then (2.16) reduces to the bi-directional Kaup-Kuperschmidt equation [18,19].

### 3.2 The $k$ - constraint of (2.3)

The  $k$ - constraint of (2.3) is given by [16]

$$L^k = B_k + \sum_{i=1}^N (q_i\partial^{-1}r_i + r_i\partial^{-1}q_i) \quad (2.18)$$

It can be seen that (2.18) together with (2.2) lead to  $L_{\tau_k} = 0$  and  $B_{n,\tau_k} = 0$ . Then (2.3) becomes  $k$ - constrained

CKP hierarchy

$$\begin{aligned} (B_k + \sum_{i=1}^N (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i))_{t_n} &= [(B_k + \sum_{i=1}^N (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i))_+^{\frac{n}{k}}, B_k + \sum_{i=1}^N (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i)], \\ q_{i,t_n} &= (B_k + \sum_{i=1}^N (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i))_+^{\frac{n}{k}}(q_i), r_{i,t_n} = (B_k + \sum_{i=1}^N (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i))_+^{\frac{n}{k}}(r_i), \quad i = 1, \dots, N \end{aligned} \quad (2.19)$$

**Example 5 (The second type of KKESCS)** For  $n = 5, k = 3$ , (2.19) presents the second type of Kaup-Kuperschmidt equation with self-consistent sources

$$\begin{aligned} u_{t_5} + \frac{1}{9}u^{(5)} + \frac{25}{6}u' u^{(2)} + \frac{5}{3}uu^{(3)} + 5u^2 u' &= \frac{1}{3} \sum_{i=1}^N [\frac{20}{3}q_i^{(3)} r_i + \frac{20}{3}r_i^{(3)} q_i + 10q_i^{(2)} r_i' + 10r_i^{(2)} q_i' + \\ &\quad 20uq_i' r_i + 20uq_i r_i' + 20u' q_i r_i], \\ q_{i,t_5} &= q_i^{(5)} + 5uq_i^{(3)} + \frac{15}{2}u' q_i^{(2)} + (\frac{35}{6}u^{(2)} + 5u^2 + \frac{10}{3} \sum_{i=1}^N q_i r_i) q_i' + [\frac{5}{3}u^{(3)} + 5uu' + \frac{5}{3} \sum_{i=1}^N (q_i r_i)'] q_i, \\ r_{i,t_5} &= r_i^{(5)} + 5ur_i^{(3)} + \frac{15}{2}u' r_i^{(2)} + (\frac{35}{6}u^{(2)} + 5u^2 + \frac{10}{3} \sum_{i=1}^N q_i r_i) r_i' + [\frac{5}{3}u^{(3)} + 5uu' + \frac{5}{3} \sum_{i=1}^N (q_i r_i)'] r_i, \\ &\quad i = 1, \dots, N \end{aligned} \quad (2.20)$$

**Example 6 (The second type of BDKKESCS)** For  $n = 3, k = 5$ , (2.19) gives rise to the second type of bi-directional Kaup-Kuperschmidt equation with self-consistent sources

$$\begin{aligned} -\frac{5}{9}u_{t_3}^{(2)} - \frac{5}{3}uu_{t_3} - \frac{5}{9}\partial_x^{-1}u_{t_3 t_3} + \frac{1}{9}u^{(5)} + \frac{25}{6}u' u^{(2)} + \frac{5}{3}uu^{(3)} - \frac{5}{3}u' \partial_x^{-1}u_{t_3} + 5u^2 u' + 2 \sum_{i=1}^N (q_i' r_i + q_i r_i') &= 0, \\ q_{i,t_3} &= q_i^{(3)} + 3uq_i' + \frac{3}{2}u' q_i, \quad r_{i,t_3} = r_i^{(3)} + 3ur_i' + \frac{3}{2}u' r_i, \quad i = 1, \dots, N \end{aligned} \quad (2.21)$$

#### 4. Conclusion

We firstly propose a new multi-component CKP hierarchy (mcCKP) based on the eigenfuction symmetry constraint for the CKP hierarchy. This mcCKP includes two types of CKP equation with self-consistent sources. It admits reductions to the  $k$ -constrained CKP hierarchy containing the second type of some  $(1+1)$ -dimensional soliton equation with self-consistent sources, and reduction of CKP hierarchy including the first type of some  $(1+1)$ -dimensional soliton equation with self-consistent sources. Thus the mcCKP provides an effective approach to find some  $(1+1)$ -dimensional and  $(2+1)$ -dimensional soliton equations with self-consistent sources and their related Lax representations. We notice that no solution has been obtained not only for the first type of CKPSCS but for the second type. So we will solve the integrable equations in the forthcoming paper.

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#### Reference

- [1] Date E, Jimbo M, Kashiwara M and Miwa T 1981 J. Phys. Soc. Japan 50 3806-3812.

- [2] Jimbo M and Miwa T 1983 Publ. Res. Inst. Math. Sci 19 943-1001.
- [3] Sato M and Sato Y 1982 Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold. Nonlinear partial differential equations in applied science (Tokyo).
- [4] Date E , Jimbo M, Kashiwara M and Miwa T 1982 Publ. Res. Inst. Math. Sci 18 1077-1110.
- [5] V G Kac and J W van de Leur 2003 J. Math. Phys. 44 3245-3293.
- [6] Johan van de Leur 1998 J. Math. Phys. 39 2833-2847.
- [7] Aratyn H, Nissimov E and Pacheva S 1998 Phys. Lett. A 244 245-255.
- [8] Liu X J, Zeng Y B and Lin R L 2007 A new multi-component KP hierarchy (Submitted)
- [9] Mel'nikov V K 1983 Lett. Math. Phys. 7 129-136.
- [10] Mel'nikov V K 1987 Comm. Math. Phys. 112 639-652.
- [11] Mel'nikov V K 1988 Phys. Lett. A 128 488-492.
- [12] Xiao T and Zeng Y B, 2004 J. Phys. A: Math. Gen. 37 7143-7162.
- [13] Wang H Y 2007 Some Studies on soliton equations with self-consistent sources. PhD thesis, Chinese Academy of Sciences.
- [14] Dickey L A 2003 Soliton equation and Hamiltonian systems (Singapore: World Scientific).
- [15] Date E , Jimbo M, Kashiwara M and Miwa T 1981 J. Phys. Soc. Japan 50 3813-3818.
- [16] I. Loris 1999 Inverse Problem 15 1099-1109.
- [17] D J Kaup 1980 Stud. Appl. Math. 62 189-216.
- [18] Dye J M and Parker A 2001 J. Math. Phys. 42 2567-2589.
- [19] Dye J M and Parker A 2002 J. Math. Phys. 43 4921-4949.